Stability of Nonlinear Systems
–An Introduction–

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The Concept of Stability

Consider the generic nonlinear system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \; \mathbf{f} : D \to \mathbb{R} \tag{1}$$

such that $\mathbf{f}(\mathbf{x})$ is continuously differentiable and locally Lipschitz, *i.e.*, $\| \mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y}) \| \leq L \| \mathbf{x} - \mathbf{y} \|$. Assume, without loss of generality, that 0 is an equilibrium point of this system, *i.e.*, $\mathbf{f}(0) = 0$.

Definition

The equilibrium point $\mathbf{x} = 0$ is said to be:

- **stable** if $\forall \, \varepsilon > 0, \exists \, \delta = \delta(\varepsilon) > 0$, s.t.

$$\| \mathbf{x}(0) \| < \delta \Longrightarrow \| \mathbf{x}(t) \| < \varepsilon, \; \forall \, t > 0 \tag{2}$$

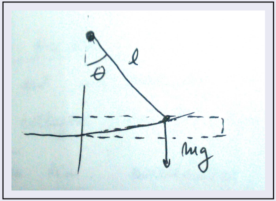
- **unstable** otherwise
- **asymptotically stable** if it is stable and δ can be chosen such that $\| \mathbf{x}(0) \| < \delta \Longrightarrow \lim_{t \to \infty} = 0$

The Concept of Stability

Observation: the above definition is a challenge-answer process: "for any value of ε, we must produce a δ that satisfies the conditions in the definition:" an impractical approach.

Example

Consider a pendulum with friction



Described by the equation: $ml\ddot{\theta} = -mg\sin\theta - kl\dot{\theta}$

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Pendulum example, continued

Coordinate change to obtain a first-order ODE: $x_1 = \theta; x_2 = \dot{\theta}$

$$\dot{x}_1 = x_2 \tag{3}$$

$$\dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 \tag{4}$$

Solutions: periodic $(n\pi, 0)$, $n = 0, \pm 1, \pm 2, \ldots$.

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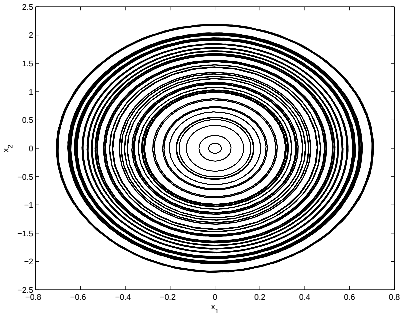
Pendulum example, continued



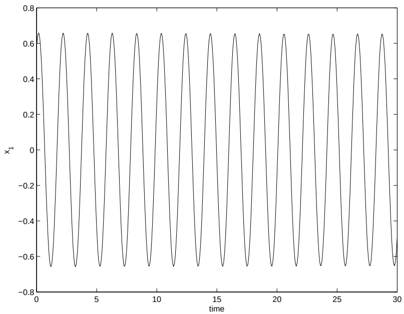


Figure: State trajectories with $g = 9.81$, $l = 0.981$, $m = 1$, $k = 0$

In the absence of friction, the system **is stable** in the sense of the definition given above.

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Pendulum example, continued

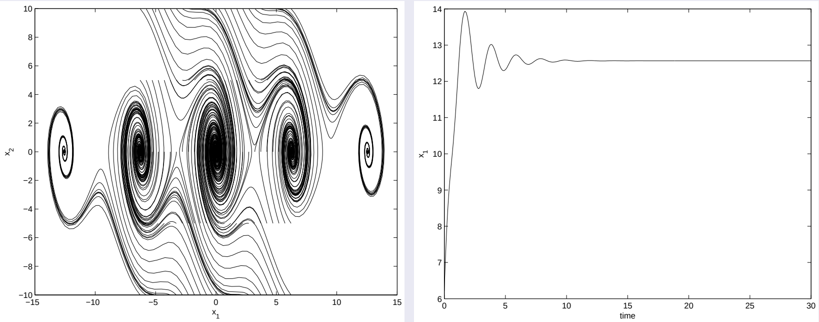


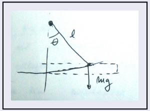
Figure: State trajectories with $g = 9.81$, $l = 0.981$, $m = 1$, $k = 1$

Friction attenuates oscillations and the pendulum eventually returns to the origin. It is therefore **asymptotically stable** in the sense of the definition given above.

An Alternative Approach

Observation: using the graphical method is cumbersome and very impractical for large(r)-scale systems ($n > 2$). Is there a more general way to establish stability?

Pendulum example, continued



Define the energy of the pendulum, $E(\mathbf{x}) = KE + PE$:

$$
\begin{aligned}
KE &= 0.5ml^2\dot{\theta}^2 && (5)\\
PE &= mgl(1-\cos\theta) && (6)\\
E(\mathbf{x}) &= \frac{1}{2}ml^2x_2^2 + mgl(1-\cos x_1) && (7)
\end{aligned}
$$

An Alternative Approach

Pendulum example, continued

- **no friction** $\Longrightarrow$ **no energy dissipation**, $E(\mathbf{x}) = c$, or $\frac{dE}{dt} = 0$ along the system trajectories. $E(\mathbf{x}) = c$ forms a closed contour around $\mathbf{x} = 0$, which is a stable equilibrium point.
- **friction** $\Longrightarrow$ **energy dissipation**, $\frac{dE}{dt} \leq 0$ along the system trajectories. Pendulum eventually returns to the stable equilibrium point $\mathbf{x} = 0$.

Examining the derivative of $E(\mathbf{x})$ along the state trajectories provides indications on the stability of the system.

Question

Is it possible to define and use some function other than energy to assess system stability?

Lyapunov Stability

Let $V : D \to \mathbb{R}$ be a continuously differentiable function. The derivative of V along the state trajectories of $\mathbf{x}$ is given by:

$$\dot{V}(\mathbf{x}) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}\dot{x}_i = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(\mathbf{x}) = \frac{\partial V}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}) \tag{8}$$

Lyapunov Stability

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Theorem (Lyapunov's stability theorem)

Let $\mathbf{x} = 0$ be an equilibrium point of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $D \subset \mathbb{R}^n$ a domain containing $\mathbf{x} = 0$. Let $V : D \to \mathbb{R}$ be a continuously differentiable function, such that:

$V(0) = 0$; $V(\mathbf{x}) > 0$ in $D - \{0\}$; $\dot{V}(\mathbf{x}) \leq 0$ in D

Then, $\mathbf{x} = 0$ is stable.

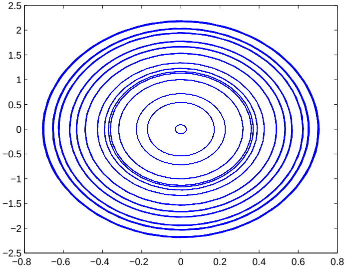
Moreover, if

$\dot{V}(\mathbf{x}) \leq 0$ in $D - \{0\}$,

$\mathbf{x} = 0$ is asymptotically stable.

Lyapunov Stability

The function V is referred to as a **Lyapunov function**. The surface $V(\mathbf{x}) = c$ is a **Lyapunov surface**.



- if $\dot{V} \leq 0$, when the trajectory crosses a surface, it will not cross back again.
- if $\dot{V} < 0$, the state trajectory crosses the surfaces with decreasing values of C, shrinking to the origin.

Observations

- The Lyapunov stability theorem can be applied without solving the ODE system
- The theorem provides a **sufficient** condition for stability
- The theorem does not provide a systematic method for constructing the Lyapunov function V of a system.

Constructing (candidate) Lyapunov functions

- energy is a natural candidate if well defined
- quadratic functions of the form $V = \mathbf{x}^T\mathbf{P}\mathbf{x}$, with $\mathbf{P}$ real, symmetric and positive (semi)definite.

Examples

Example 1: Pendulum Energy as Lyapunov Function

Consider pendulum without friction. We have

$$V(\mathbf{x}) = mgl(1 - \cos x_1) + 0.5ml^2x_2^2 \tag{9}$$

We can verify that:
$V(0) = 0$ and $V(\mathbf{x}) > 0$ for $-2\pi < x_1 < 2\pi$.

$$\begin{aligned}
\dot{V}(\mathbf{x}) &= mgl\dot{x}_1 \sin x_1 + ml^2 x_2 \dot{x}_2 && (10)\\
&= mglx_2 \sin x_1 + ml^2 x_2 \dot{x}_2 && (11)\\
&= mglx_2 \sin x_1 + ml^2 x_2 [-\frac{g}{l} \sin x_1] && (12)\\
&= mglx_2 \sin x_1 - mglx_2 \sin x_1 && (13)\\
&= 0 && (14)
\end{aligned}$$

Conclusion: Pendulum is stable (**not** asymptotically stable).

Examples

Example 2: A Quadratic Lyapunov Function for Pendulum with Friction

Will be worked in class.

Conclusions/Food for Thought

- Lyapunov theory: Powerful framework for establishing the stability of **any** dynamical system without the need for an explicit solution
- Translates naturally to linear systems
- Extension to non-autonomous nonlinear systems, input-to state stability
- Lyapunov-based nonlinear controller synthesis
- Only sufficient condition: need to define and test Lyapunov function candidate
- Energy: central role (think large-scale systems/networks).